The function of functions

An approach to Walsh functions from telecommunications history

by Thomas Roddam

Named after their originator, an American mathematician, Walsh functions are now beginning to find applications in electronics. This article first discusses the use of mathematical functions in general in telecommunications then goes on to illustrate the nature of Walsh functions through a practical technique for avoiding crosstalk between overhead telephone wires. Generation of Walsh functions and some of their applications will be dealt with in the concluding part of the article to be published later.

At somewhat irregular intervals readers of *Wireless World* find themselves confronted by an article on some mathematical function. It may be, indeed it often is, our old friend the exponential, or it may be, say, Muratori's function. Why does this happen, why do we write these things, why do you read them?

It is not just the money, barely enough to pay the ink bill, which makes the author produce this stuff. There is a real satisfaction in attempting to make poor old $\exp(x)$ fresh and interesting: there is a real challenge in explaining Muratori's function clearly without boring the reader stiff.

The reader is more of a problem. Many years ago the editor, not this one or his predecessor, told me how he had actually seen a reader, reading the latest issue. In the Underground. However, little is known about the great mass who live a no doubt quiet and industrious life, and never write letters or complete questionnaires. The problem is quite simply this. Either they know all about the Binomial Theorem, let us say, or they don't. If they don't, either they need to, or they don't. The last group have lived happily in ignorance, while the ignorant who need to know must surely need to know more than can be packed into a few pages.

The answer, I have decided, lies in the sort of people we are. In most organisations there are two sets of people. There are the hard-headed men committed to getting stuff out of the factory gate and the long-haired boys messing about with slide-rules. If you prefer it there are the fossils who spend a week getting it wrong with a soldering iron rather than a morning on the computer finding an optimum solution. Muratori's function is a weapon used by the theorist to defend himself against the pragmatist, especially if the pragmatist is his boss. Know your enemy.

With this in mind I began to peer back into the early days of our trade. It turns out that we have been in business longer than I thought. The electric telegraph is, of course, the starting point, but it is surprising to find that the proposal for an electric telegraph actually preceded the work of Volta and Galvani. The first proposal, in the *Scots Magazine*, was in 1753, and the scheme was to use 26 wires, each with a hanging pith ball which would strike a bell, using a Leyden jar as source. Once the cell had been invented, and Oersted had found that a current would influence a magnet, the way was open.

By about 1850 things were really moving and the contrasts, the tunnel vision, all the factors of our modern technology were showing themselves in all their glory. The submarine cable, and especially the Atlantic cable, bring out all that is finest in pragmatism, theory, and the use of theory for analysis but not for synthesis. Fig. 1

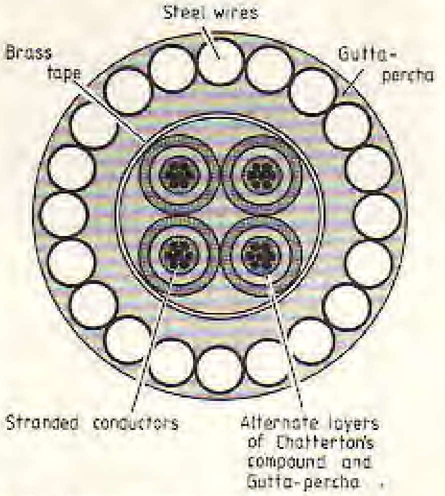

Fig 1. Cross-section of a submarine telegraph cable, as constructed at about the turn of the century.

comes from Notes on Telegraphy, A. G. Pratt and G. Magg, which my mother seems to have bought in 1903. The use of the stranded conductor was the idea of Professor William Thomson, later Lord Kelvin, in 1854. Clearly he was a sound practical man. In 1855, however, he was considering the partial differential equation

$$LC\frac{\partial^2 n}{\partial t^2}+(CR+LG)\frac{\partial n}{\partial t}+RGn=\frac{\partial^2 n}{\partial x^2}$$

The trouble is that he decided to neglect the inductance, L, and the leakage, G. The full equation, called the telegrapher's equation, was published by Kirchhoff in 1857, and forgotten, by Heaviside in 1876, but Heaviside never had any luck, and by Poincaré in 1893. Thomson comes up with a solution for the line current at time t, I_t, in terms of the maximum current the battery can produce, I_o, of:

$$I_t=I_o(1-2(\epsilon^{-\pi 2t/kcl2} -\epsilon^{-4\pi 2t/kcl2}+\epsilon^{8\pi 2t/kcl2}\ldots))$$

where $\epsilon=(3/4)^{t/a}$ and $a=kcl^2\log_e(4/3)/\pi^2$

There's glory for you. At the end of the day it boils down to saying that for a particular type of line the speed of working is inversely proportional to the square of the length.

At this point there are three ways to go. The first, Thomson again, is the purely instrumental one. When the battery is applied at one end of the great distributed RC circuit the current starts to grow, very slowly, at the far end. Invent a very sensitive detector and it will only be necessary to hold the key down for a relatively short time to get a signal, and the reduced charge in the system will soon die away ready for the next mark.

The next step is to use what politicians call a U-turn: at the end of a positive mark the battery is reversed, to send a curbing current down the line. The duration of the curbing current was changed according to the speed of working but was typically about four-fifths of the mark pulse. After the curb came an inter-pulse interval, with the line earthed.

This is nothing but something we tend to regard as quite a modern idea. The signal characteristics have been tailored, coded, to suit the characteristics of the medium. Indeed, the telegraphers did quite a lot of this. Morse produced a code in which the commonest letters used the shortest groups, and on the long cables, with the sensitive receivers, input and output capacitors were used to eliminate the effects of earth currents. Then they went to multiplexing by using three-value logic, and to some quite sophisticated time division multiplex systems for short lines, with synchronisation between the two ends.

All this ingenuity, all this tedious calculation of the rise and fall of current in long lines, but no-one really looking at the telegrapher's equation. At least, memory suggests that Heaviside did, but his sad cry 'even Cambridge mathematicians deserve justice' summarizes his influence. In Europe the invention of the loading coil is attributed to Pupin, but really it is sitting there, just waiting for someone to ask "what value of L do we need?"

If there is a moral, and I think there is one, it is that it is a waste of time to use mathematics to find out why it works. Use the mathematics to find out if it will work, or how to make it work better.

Under certain conditions the telegrapher's equation brings up the Bessel functions in its solutions. The Bessel functions weave in and out of the history of telecommunications. They became very trendy just after someone had the idea of sticking a paper cone to the centre of an ear-piece, instead of fastening the ear-piece to the end of a large horn. Looking back we can ask why there was such interest in calculating how the cone would break up into spatial harmonics when the real problem was to prevent this happening at all. More recently the Bessel functions have appeared in filter design, although I found them in a pulse response problem quite a long time ago.

Then, of course, there was frequency modulation. The idea, that by keeping the carrier going at full power all the time the noise at the receiver could be kept down, seems a fair one to use for examining a system. And it seemed to work. The theoreticians began to study the characteristics of

$e=E_o\sin(\omega t+m_f\sin pt)$, where

$\omega=2\pi f_c$, with f_c the centre frequency

$p=2\pi f_s$, with f_s the signal frequency

and m_f, the modulation index, is the ratio $\delta f_c/f_s$.

When this expression is expanded it becomes

$$e=E_o[J_o(m_f)\sin\omega t$$
$$+J_1(m_f)\ [\sin(\omega+p)t-\sin(\omega-p)t]$$
$$+J_2(m_f)[\ \ldots\ (\omega+2p)\ \ldots\ (\omega-2p)$$
$$+J_3(m_f)\ \ldots$$

At this point the interpreters did the wrong thing. If the spectrum is to be kept into the same bandwidth as we need for amplitude modulation we must have $J_2(m_f)$ and the higher Bessel functions small, so that the $(\omega+2p)$, $(\omega+3p)$ etc. terms can be neglected. This leads to a modulation index of about one half, for which the J_2 term becomes about 3%. If you go on to calculate the noise advantage you find that the whole thing is just a lot of nonsense. Mathematically it is clear that there is no point in taking it seriously. Every schoolboy knows now that the two keys to f.m. operation are hard limiting and a high modulation index.

Here we have the theoreticians saying something would not work, and the practical man showing that it did. A rather bizarre phase was the 'sidebands don't exist' period. The expansion of

$$A(1+m\sin 2\pi f_s t)\sin 2\pi f_c t$$

to give a carrier, $A\sin 2\pi f_c t$, and two sidebands at $(f_c\pm f_s)$, is not the most difficult mathematics we expect to meet. It was, however, too much for a school of thought, still alive around 1930, which held that the signal was there, in the carrier, and could be received with a very narrow band receiver. Circuits were published, sets were made. We shall never know just why they seemed to work, but there are two obvious possibilities. The narrow bandwidth was produced by a string of tuned circuits, which would not be all that narrow even if they were tuned to the same frequency. The detectors used

Sir George Jefferson, chairman of British Telecom, waves cheerily from an elevated position at BT's training school, where engineers practise climbing on these short poles.

then behaved much better at low modulation, so that the carrier enhancement would have improved the detector. The audio amplifier, with *CR* interstage coupling, could easily have boosted up the lost treble. Alternatively, or additionally, we must not forget one of the great design problems of the time, the feedback from anode to grid through the valve capacitance. Strong coupling, both capacitive and inductive, between the tuned circuits must have been present. Immediately we have a bandpass structure, not a single narrow slit. The true believers would not be deterred.

I referred to this as a bizarre event, because it took place when multi-channel carrier systems were already in use on telephone lines. The distance-limit of speaking by telephone depends on the product of the resistance of the circuit, (in ohms) *R*, and the capacitance of the circuit (in microfarads) *K* – or *KR*. The following figures show approximately the *KR* which limits easy and practical speech, and indicate the telephonic value of the conductors:

copper wire (open)	*KR* 10,000
cables or underground lines	8,000
iron wire (open)	5,000

The low value of iron is due to the presence of *electromagnetic inertia*, which is absent in copper.

So the next step was to put in more electromagnetic inertia, in the form of the loading coil.

The great influence which the loading coil was to have on the communications industry arose from the simple fact that the numbers needed were enormous. In the Bell System light loading was a coil every 6,000ft, and heavy loading a coil every 3,000ft. At 3,000Hz loading brought the attenuation per loop mile down from about 2dB to about 0.5dB. Longer circuits, better circuits, more traffic, and so more circuits and more loading coils. The size and the spacing demanded close study. This study, of a long ladder of series inductors and shunt capacitors, brought the functions cosh θ and sinh θ into the communication engineer's life. The development of the low-pass filter, followed by the other classic filters, from the long line analysis explains the awkwardness of early filter theory. In the long line the problems of end effects were relatively trivial, but the ends could wag the filter if only a couple of sections sufficed. Clever systems of high class bodging, like m-derivation, mm' derivation, α-matching, and tedious calculations of mis-match and interaction loss made filter disign an art. Then we found Tchebycheff. If my memory is correct, his interest, in St Petersbourg (he wrote in French) in 1875, was steam engines. All those shiny bits that move to and fro, while the wheels go round, should move in a straight line. Like the pass-band response of a filter. The Tchebycheff functions were a step in linkage design.

Not very much relevant to our theme can be found in the history of modern filter design. Once it was seen that the problem was, quite simply, to design a finite network of defined properties, it became a matter of using well-known techniques. The vital step was the realisation that the idea was to find the best value to use in the structures which had grown up from the long line.

Softly the functions come and go, or, if your taste is more demotic, I go, I come back. The Laguerre polynomials have cropped up again, though I haven't seen them around since I dealt with a chain of regulating repeaters, back in about 1950.

The story began with telegraphy, with signals which were either marks or spaces, and moved on to telephony, with the signals a mixture of sine waves. In the 1930s, however, Alec Reeves was building one pulse modulation system after another. Before any of them came into service the digital computer was on the way. The Boolean algebra, which we had come to associate with the use of mathematics in cleaning up classical logic, began to be a really bread and butter affair.

Although Boole's logic, and the techniques based on it, like the Karnaugh map, were central to the signal processing operation, the signal frequently needed to be transmitted from place to place. The available telephone channels, and the general thinking of the radio circuit designers, were based on bandwidth, on the available chunk of frequency spectrum. Information theory, which started well before it really mattered, defined what could be done. Fourier analysis could be used to discover just what the circuits did to the pulses. There is a faint memory of Heaviside here. The pulse gives an infinite series, and then the bandwidth limitations just chops off most of the terms. In pulse modulation systems, indeed, the sine wave really needs an infinite number of pulses, and the pulses need an infinite Fourier series.

The pulse-makers clearly need a new kind of series, to do for them what the Fourier series had done for sinusoidal waveforms. It is to the favourite in this field that we now turn our attention. The biggest advance since sliced bread, we are told, is the Walsh functions, although I regard sliced bread as a cruel and unnatural punishment. But Walshites have written:

"We may well come to the point of view that if Walsh functions had been with us from the start and someone had then come up with the idea of sinusoids we would all want to know what use they were."*

A fund is being started to buy ocarinas for supporters of this view.

We have already seen how important it is to keep one's feet firmly planted on the ground when considering the use of mathematics. It is therefore appropriate to look at Fig. 2. When telegraph poles began to be used for telephone circuits it was soon found that if the two wires of one pair simply ran parallel to the two wires of another, the mutual inductance produced cross-talk from one to another. A simple answer is to split the run in half, and cross one pair at the mid point. We can write this symbolically as:

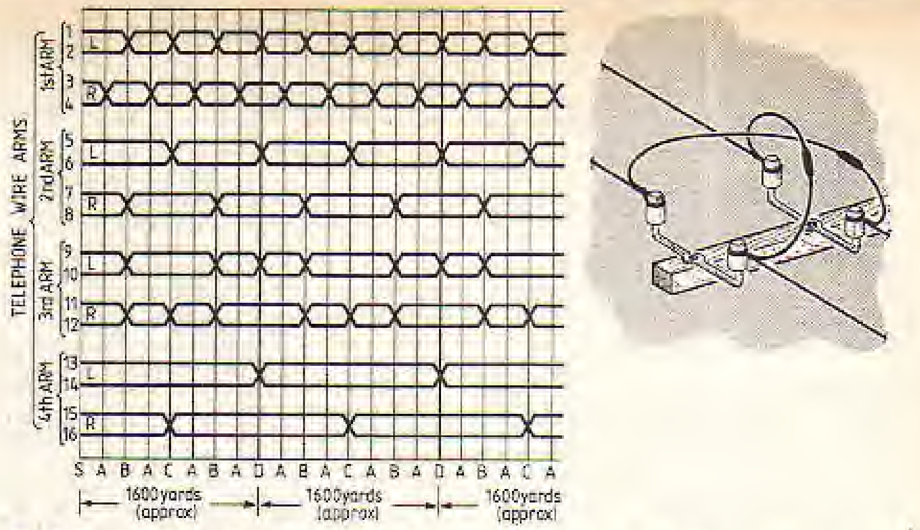

Fig. 2. Transposition of telephone wires for avoiding crosstalk caused by mutual inductance. On the left is the pattern employed and on the right the method of wiring at a transposition point. (Adapted from Railway Signalling and Communications, Tattersall et al, 1946.)

$$\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$$

When there are more than two pairs we can start by taking two pairs as a quad, and use the same symbolic solution, which we can bracket up to be a matrix:

$$\begin{pmatrix} Q & Q \\ Q & -Q \end{pmatrix}$$

This is short for:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Four pairs can be transposed according to this pattern, with the total run split into four sections. If we call this (G), we can transpose eight pairs according to the scheme

$$\begin{pmatrix} G & G \\ G & -G \end{pmatrix}$$

We can go on expanding in this way, and what we are doing is working with Hadamard matrices. Using the definition

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we have

$$H_N = H_{N/2} \otimes H_2$$

where $\otimes$ is the Kronecker product, so that

$$H_8 = H_4 \otimes H_2$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

The working of Fourier analysis depends on the fact that the sine and cosine wave system is orthogonal, so that

$$\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = \theta \text{ if } m \neq n$$

The rows, and the columns, of the Hadamard matrix have this orthogonality characteristic, which is why row 1 transposition does not couple to any other row. And the rows are, quite simply, the Walsh functions. There is another way of producing them, which gives a different order. The Rademacher functions are defined as

$$r_n(\theta) = \text{sign of } (\sin(2^{n-1}\pi\theta)),\ 0 \leq \theta \leq 1$$

and some of the Walsh functions are

$$\begin{aligned} \text{wal}\,(1,\theta) &= r_0(\theta) \\ \text{wal}\,(3,\theta) &= r_1(\theta) \\ \text{wal}\,(7,\theta) &= r_2(\theta) \\ \text{wal}\,(2^k-1,\theta) &= r_{k-1}(\theta) \end{aligned}$$

The way in which the rest of the family is derived depends on an equation which looks very simple:

$$\text{wal}\,(i,\theta)\,.\,\text{wal}\,(j,\theta) = \text{wal}\,(i \oplus j, \theta)$$

The symbol $\oplus$ stands for modulo-2 addition, which is binary addition without a carry sign. If we take

$$\begin{aligned} 1 &\rightarrow 0001 \\ \oplus 3 &\rightarrow \underline{0011} \\ 2 &\leftarrow 0010 \end{aligned}$$

so that wal $(1,\theta)$. wal $(3,\theta)$ = wal $(2,\theta)$

★ R. Barrett, J. A. Gordon, D. Brammer. Theory and applications of Walsh functions. Hatfield Polytechnic Symposium 1971.

† I am indebted to Mr A. Emmerson of British Telecom for locating Fig. 2 in the book referred to.

A set of wal functions is shown as Fig. 3. A point to notice is that θ is a time base,

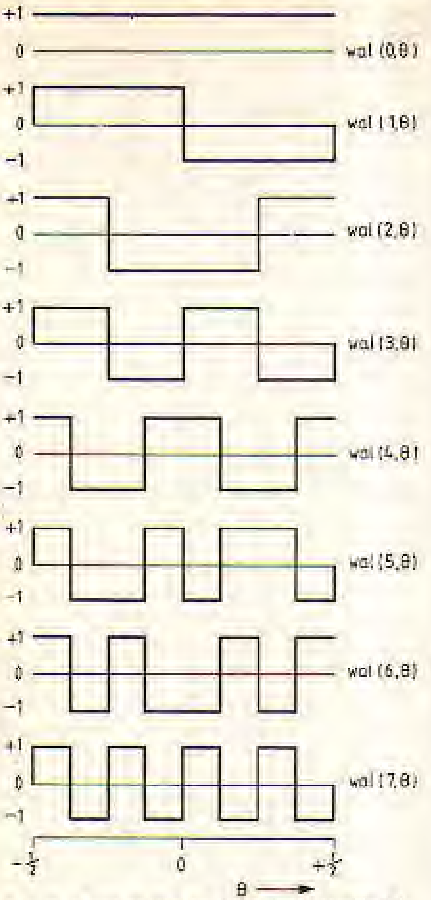

Fig. 3. A set of Walsh functions, wal(n,θ). Note that θ is a time base and that, as the functions have the values ±1, they are rectangular in form.

which goes from −½ to +½ in the time interval T. Another important feature is that the functions can be sorted out into two groups. If you imagine a sine wave and a cosine wave which have been clipped right down, a technique used, with 20dB of clipping, for some transmission systems on noisy circuits, you will see that wal (1, θ) looks very much like a clipped sine wave, and wal (2,θ) like a cosine wave, The odd Walsh functions, which are antisymmetric, are written as sal (i, θ), while the symmetric properties of the even functions give them the form cal (i, θ).

The sine wave we assumed to be clipped right down to give sal (1,θ) possessed the property of having a frequency. sal (1, θ), a single cycle in the sine wave, has two crossings of the zero axis in each unit of time. (As shown the end zeros are shared with the next cycle.) The *sequency* of a Walsh function is similarly defined as:

Sequency in crossings per second = ½ (average number of zero crossings per unit time)

What have we now got? A set of orthogonal functions, and the concept of sequency. It is the switching man's equivalent of the sinusoids and the concept of frequency.

To be concluded in the next article, which will show how Walsh functions can be produced by hardware and discuss their use.