Applying the Analog Differentiator

DON HOFT, Senior Engineer, Raytheon Company, Wayland, Mass.

Donald J. Hoft is a Senior Engineer and Design Group Leader with the Raytheon Company's Engineering Development Laboratory, Wayland, Mass. He received his B.S.E.E. from Pratt Institute in 1959 and his S.M.E.E. from MIT in 1966.

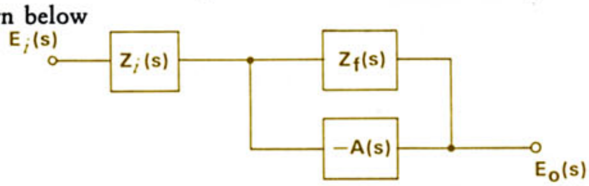
INTRODUCTION

Many engineers avoid the use of analog differentiators because of their inherent noise problems and their tendency to oscillate. Often however, particularly at low frequencies and where low noise is not a prime design objective, these difficulties may be overcome and the analog differentiator employed as a useful and often economical means of implementing a circuit requirement.

This paper analyzes the basic analog differentiator operation and discusses the limitations imposed by finite operational amplifier gain and bandwidth. Means for minimizing instabilities and noise at high frequencies are discussed as well as the requirements for obtaining improved rise time performance.

ANALYSIS

The basic circuit configuration for a feedback amplifier is shown below



The closed loop gain of this circuit is given by*

$$\frac{E_0(s)}{E_i(s)} = -\frac{Z_f(s)}{Z_i(s)}\left[\frac{1}{1+\frac{1}{A(s)\beta(s)}}\right]$$

If the gain, A, is very high, then $A(s)\beta(s) \gg 1$, and the closed loop gain is very nearly equal to

$$\frac{E_0(s)}{E_i(s)} \cong -\frac{Z_f(s)}{Z_i(s)}$$

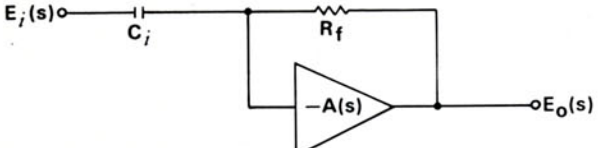
For the general case, however,

$$\beta(s) = \frac{Z'(s)}{Z'(s)+Z_f(s)}$$

Where $Z'(s)$ is the parallel combination of $Z_i(s)$ and R_d, the input impedance (in this case assumed resistive) of the amplifier. Then

$$\frac{E_0(s)}{E_i(s)} = -\frac{Z_f(s)}{Z_i(s)}\left[\frac{1}{1+\frac{1}{A(s)}\left(1+\frac{Z_f(s)}{Z'(s)}\right)}\right]$$

For a differentiator, $Z_f(s)$ is a resistor R_f, and $Z_i(s)$ becomes a capacitive reactance.



In the usual case (especially with FET-input amplifiers) $R_d \gg X_{C_i}$, therefore

$$Z'(s) = \frac{R_d\left(\frac{1}{sC_i}\right)}{R_d+\frac{1}{sC_i}} \cong \frac{1}{sC_i} = Z_i(s)$$

*"Operational Amplifiers, Part 1," Analog Devices, Inc., Applications Manual.

and

$$\frac{E_0(s)}{E_i(s)} = -sR_fC_i\left[\frac{1}{1+\frac{1}{A(s)}(1+sR_fC_i)}\right]$$

For an operational amplifier, the gain as a function of frequency is given by

$$A(s) = \frac{A_0}{1+T_0s}$$

where A_0 is the low frequency gain, $1/T_0$ is the (radian) frequency breakpoint, and $\omega_c = A_0/T_0$ is the unity gain frequency (Figure 1).

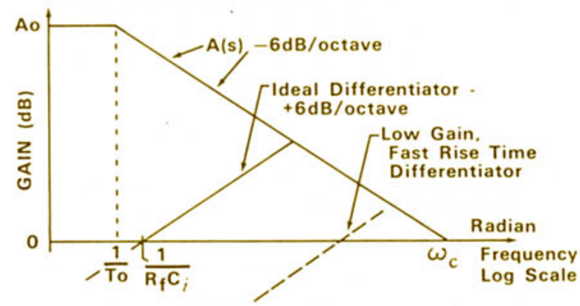


Figure 1. Bode Plot of Ideal Differentiators

Then

$$\text{(1)} \quad \underbrace{\frac{E_0(s)}{E_i(s)} = -sR_fC_i}_{\text{IDEAL DIFFERENTIATOR}}\underbrace{\left[\frac{1}{1+\left(\frac{1+T_0s}{A_0}\right)(1+sR_fC_i)}\right]}_{\text{ERROR TERMS}}$$

The first term is the ideal response, while the second term, in brackets, is the error caused by finite gain and bandwidth. If the gain, $A_0 \to \infty$, then we have the ideal differentiator characteristic (gain increasing with frequency) shown plotted in Figure 1. Typically the operating frequency range of a differentiator is around the frequency at which the gain is 0dB. From the figure, it is apparent that (1) highest gain is obtained at high frequencies, thus making the differentiator susceptible to high frequency noise and, (2) an unstable condition is produced because the ideal differentiator function (+6dB/octave) closes on the gain curve (−6dB/octave) at 12dB/octave (i.e. with 180° phase shift). These problems, which are readily overcome, are discussed later. However, valuable insights into a differentiator's extreme high speed behavior may be obtained by computing its time response with the idealized feedback network. Rearranging equation (1) while noting that $1/A_0 \cong 0$, gives

$$\frac{E_0(s)}{E_i(s)} = -s\omega_c\left[\frac{1}{s^2+s\left(\frac{1}{R_fC_i}+\frac{1}{T_0}\right)+\frac{\omega_c}{R_fC_i}}\right]$$

We are now in a position to "complete the square" to obtain the Laplace Transform in the "standard" form

$$\frac{1}{(s+a)^2+b^2}$$

This operation yields (2) $\frac{E_0(s)}{E_i(s)} =$

$$-s\omega_c \left[\frac{1}{\left(s + \left[\frac{1}{2T_0} + \frac{1}{2R_fC_i}\right]\right)^2 + \left(\frac{\omega_c}{R_fC_i} - \left[\frac{1}{2T_0} + \frac{1}{2R_fC_i}\right]^2\right)} \right]$$

Before proceeding to obtain the inverse transform of this equation, we find that, for typical values of parameters in operational amplifier differentiators,

(3) $$\frac{\omega_c}{R_fC_i} \gg \left(\frac{1}{2T_0} + \frac{1}{2R_fC_i}\right)^2$$

With this simplification, equation (2) becomes

(4) $$\frac{E_0(s)}{E_i(s)} \cong -s\omega_c \left[\frac{1}{\left(s + \left[\frac{1}{2T_0} + \frac{1}{2R_fC_i}\right]\right)^2 + \frac{\omega_c}{R_fC_i}} \right]$$

This is the £-transform equation of the basic analog differentiator, taking into account the amplifier's open-loop gain characteristic.

RAMP INPUT

To illustrate the performance of the differentiator, consider the response of (4) to a linear ramp input (i.e., a signal whose derivative steps from zero at the beginning of the ramp to a constant value, k)

(5) $$e_i(t) = kt \qquad (k = \text{constant})$$
or
$$E_i(s) = \frac{k}{s^2}$$

Equation (4) then becomes

$$E_0(s) = -\frac{k\omega_c}{s} \left[\frac{1}{\left(s + \left[\frac{1}{2T_0} + \frac{1}{2R_fC_i}\right]\right)^2 + \frac{\omega_c}{R_fC_i}} \right]$$

Taking the inverse transform yields

$$e_0(t) = -\frac{k\omega_c}{(\omega_c/R_fC_i)^{1/2}} \int_0^t e^{-\left(\frac{\omega_c}{2A_0} + \frac{1}{2R_fC_i}\right)t} \sin\left(\frac{\omega_c}{R_fC_i}\right)^{1/2} t \, dt$$

Performing the integration and noting the inequality (equation 3) yields the output given by (6) and depicted in Figure 2.

(6) $$e_0(t) = kR_fC_i \left[e^{-\left(\frac{\omega_c}{2A_0} + \frac{1}{2R_fC_i}\right)t} \cos\left(\frac{\omega_c}{R_fC_i}\right)^{1/2} t - 1 \right]$$

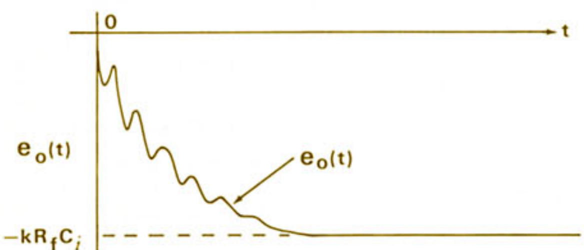


Figure 2. Ramp Response of "Ideal" Differentiator

Considerable insight into differentiator performance may be obtained from equation (6) and Figure 2 for the basic circuit. As expected, the differentiator output response to a linear ramp is seen to be a DC level proportional to the magnitude, k, of the input ramp. The gain of the circuit is equal to $-R_fC_i$, while the rise time is a function of the operational amplifier parameters (ω_c and A_0) as well as the external circuit elements (R_f and C_i). The superimposed damped sinusoid during the rise time is rarely in evidence in a practical differentiator, since the ringing is almost always above the cutoff frequency of the practical circuit (as discussed below). In the majority of analog differentiator circuit designs, where rise time is not a prime consideration, the gain, $-R_fC_i$, is often set between 0.1–1.0, in which case

$$\frac{\omega_c}{2A_0} \gg \frac{1}{2R_fC_i}$$

and the rise time is almost completely determined by the op amp parameters. If one is designing for an application in which fast rise time is an important consideration, then, from equation (6), it is evident that faster rise times may be obtained by selecting R_f and C_i, (shown in Figure 1) as small as possible. This improvement in τ_r is obtained, however, at the expense of circuit gain (gain = $-R_fC_i$), which may, of course be recouped by following the differentiator with a wideband amplifier. (The small output amplitude will also have less tendency to tax the slew rate capability of the amplifier.)

If the differentiator is being utilized to measure the slope of the input ramp, the accuracy of measurement at a time t_1 is seen to be very nearly

$$\text{Accuracy (\%)} \cong 100\, e^{-\left(\frac{\omega_c}{2A_0} + \frac{1}{2R_fC_i}\right)t_1}$$

PRACTICAL CONSIDERATIONS

Earlier it was pointed out the basic analog differentiator is susceptible to high-frequency noise amplification and instabilities (due to capacitance to ground at the input and output of the amplifier in practical circuits, and to greater than 6dB/octave rolloff slope in many available amplifiers). A common technique utilized to improve circuit performance with respect to both problems is the addition of a single or double 6dB/octave "breakpoint," as shown in Figures 3 and 4.

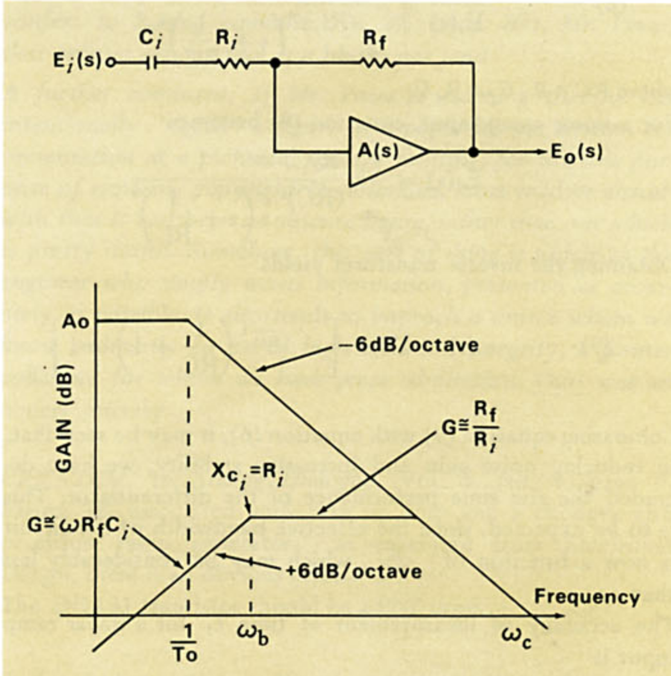


Figure 3. Bode Plot of Differentiator with First-order Rolloff

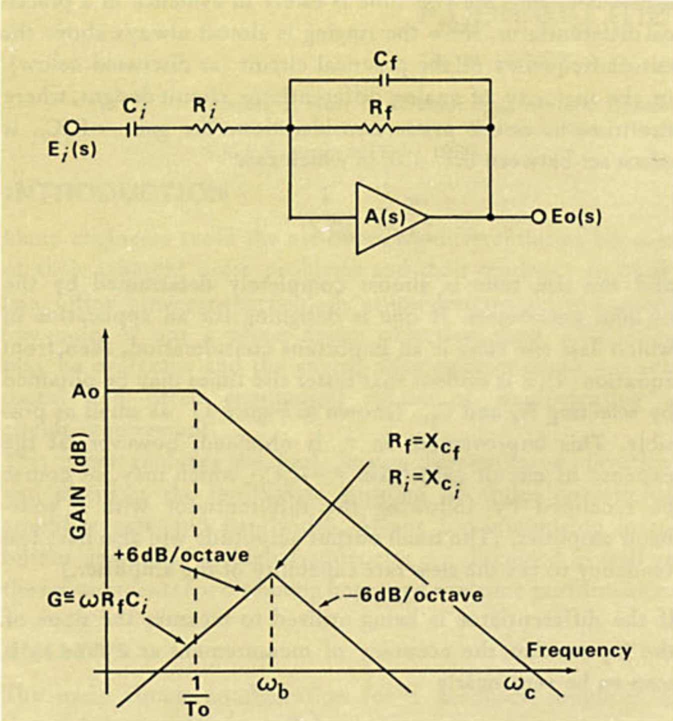


Figure 4. Bode Plot of Differentiator with Second-order Overdamped Rolloff

In either case, the supplementary breakpoints should be placed well enough above the maximum operating frequency to assure the required accuracy.

The expression for the transfer fraction of the circuit of Figure 4 is approximately

$$(7) \qquad \frac{E_0(s)}{E_i(s)} \cong - \frac{sR_fC_i}{(1 + sR_iC_i)(1 + sR_fC_f)}$$

If $R_f = X_{C_f}$ and $R_i = X_{C_i}$ at ω_b, equation (7) becomes

$$(8) \qquad \frac{E_0(s)}{E_i(s)} = - \frac{R_fC_i}{(RC)^2} \frac{s}{\left(s + \frac{1}{RC}\right)^2}$$

where $RC = R_f C_f = R_i C_i$

For a linear ramp input, equation (8) becomes

$$E_0(s) = -k \frac{R_fC_i}{(RC)^2} \frac{1}{s\left(s + \frac{1}{RC}\right)^2}$$

Obtaining the inverse transform yields

$$(9) \qquad e_0(t) = kR_fC_i \left[e^{-\frac{1}{RC}t} \left(\frac{t}{RC} + 1 \right) - 1 \right]$$

Comparing equation (9) with equation (6), it may be seen that, in reducing noise gain and increasing stability, we have degraded the rise time performance of the differentiator. This is to be expected, since the effective bandwidth of the circuit is now a function of $\frac{1}{RC}$, which may be considerably less than ω_c.

The accuracy of measurement at time t_1 for a linear ramp input is

$$\text{Accuracy} \cong 100\, e^{-\frac{1}{RC}t_1} \left(\frac{t_1}{RC} + 1 \right)$$

Editor's Notes:

Your response to *Analog Dialogue* and the articles we publish continues to encourage us, and to give us the heartwarming feeling that not only is the *Dialogue* important to our business, but also that we have been charged with a responsibility for the nurture of an organism whose life now has an existence of its own.

Since the last issue, much has happened at Analog Devices, with a direct bearing on the future of *Dialogue*. What it boils down to is that although our field remains the same ("Analog Technology"), our product line is now greatly expanded, with greater depth (fine structure) in a number of areas. In future issues, one can hope to read about

- Active Filters
- New Linear Integrated Circuit Designs & Applications
- Power Amplifier Applications
- System Design Using A/D and D/A Converters
- New types of Converters Using Monolithic IC's
- Analog Multiplier Specifications & Applications

Despite a number of past long silences, we may also hope to see *Dialogue* more frequently in the future.

ANALOG UPDATE. You may have noticed, in recent months, the appearance in our mailings of what seemed a poor substitute for *Dialogue*, a one-page summary of new products called *Analog Update*. Here's what *Update* is all about:

It is the technological mission of Analog Devices to identify and produce electronic function circuits having the performance you need at prices that will enable you to use them profitably in your own circuit, instrument, and system designs, and to provide you with as much as possible in the way of useful design information and applications assistance to make their use easy. To accomplish this mission requires that we communicate news of new products (or applications of existing products) to you in a timely fashion.

In the not-too-distant past, it was easy to communicate new product applications ideas because the number of new products introduced was relatively small—and they were all operational amplifiers! However, as we've grown, the number and complexity of products has greatly increased. For example, during 1969 we introduced about 20 new products, including such diverse items as Discrete and IC Operational Amplifiers, Power Supplies for Op Amps and Logic circuits, Log Circuits, an analog circuit "Solderless Breadboard," Instrumentation Amplifiers, Active Filters, etc.

To fulfill the need to communicate new product information frequently, we have initiated *Analog Update* to serve "between the *Dialogues*" as a rapid-access medium to inform you of new Analog Devices products. *Update* will continue to be brief and to the point, it will usually be accompanied by product data sheets, or other timely material, and it will appear aperiodically (but frequently!).

DAN SHEINGOLD